A variation of multiple L-values arising from the spectral zeta function of the non-commutative harmonic oscillator

Kazufumi KIMOTO* and Yoshinori YAMASAKI†

May 8, 2008

Abstract

A variation of multiple L-values, which arises from the description of the special values of the spectral zeta function of the non-commutative harmonic oscillator, is introduced. In some special cases, we show that its generating function can be written in terms of the gamma functions. This result enables us to obtain explicit evaluations of them.

Keywords: Multiple zeta values, multiple L-values, Bernoulli numbers, non-commutative harmonic oscillator, spectral zeta function, symmetric functions.

2000 Mathematical Subject Classification: 11M41, 05E05.

1 Introduction

The multiple zeta values

$$\zeta_k^{\bullet}(n_1,\dots,n_k) := \sum_{1\le i_1<\dots<i_k} \frac{1}{i_1^{n_1} i_2^{n_2}\dots i_k^{n_k}} \tag{1.1}$$

are natural extensions of the Riemann zeta value $\zeta(n)=\sum_{i=1}^{\infty} i^{-n}$ introduced by Euler, and have been of continuing interest to many mathematicians [18]. Recently, it has been shown by several authors that they appear in various fields in mathematics such as the knot invariant theory, quantum group theory and mathematical physics (see, e.g. [11, 20]). This fact implies the richness of the theory of the multiple zeta values and encourages the recent studies of them. One of the main problems in studying multiple zeta values is to clarify the $\mathbb{Q}$-algebra structure of the space spanned by them, which is closely related to that of the category of mixed Tate motives. In fact, for this purpose, a plenty of results concerning relations among them and exact calculations of them are investigated. Furthermore, as a natural generalization, Arakawa and Kaneko [2] introduce two kinds of multiple L-values

$$L_{\text{ш}}(n_1,\dots,n_k;f_1,\dots,f_k) := \sum_{m_1>\dots>m_k>0} \frac{f_1(m_1-m_2)\dots f_{k-1}(m_{k-1}-m_k)f_k(m_k)}{m_1^{n_1}m_2^{n_2}\dots m_k^{n_k}}, \tag{1.2}$$

$$L_{*}(n_1,\dots,n_k;f_1,\dots,f_k) := \sum_{m_1>\dots>m_k>0} \frac{f_1(m_1)f_2(m_2)\dots f_k(m_k)}{m_1^{n_1}m_2^{n_2}\dots m_k^{n_k}}, \tag{1.3}$$

where $f_1,\dots,f_k$ are $\mathbb{C}$-valued periodic functions on $\mathbb{Z}$ and also study their relations and exact evaluations.

In this paper, we study the following variation $S_k^{(N,M)}(n_1,\dots,n_k)$ $(N,M\in\mathbb{N})$ of the multiple L-values;

$$S_k^{(N,M)}(n_1,\dots,n_k) := \sum_{1\le i_1\le i_2\le\dots\le i_k} \varepsilon^{(N)}_{i_1 i_2\dots i_k} \frac{\omega_M^{i_1+i_2+\dots+i_k}}{i_1^{n_1}i_2^{n_2}\dots i_k^{n_k}}, \tag{1.4}$$

*Partially supported by Grant-in-Aid for Young Scientists (B) No. 20740021
†Partially supported by Grant-in-Aid for JSPS Fellows No. 19002485

where ω_M is a primitive Mth root of unity and

$$\varepsilon_{ij}^{(N)} := \begin{cases} 0 & i = j \not\equiv 0 \pmod N \\ 1 & \text{otherwise} \end{cases} = 1 - \delta_{ij}\left(1 - \frac{1}{N}\sum_{r=0}^{N-1}\omega_N^{ri}\right), \qquad \varepsilon_{i_1 i_2 \dots i_k}^{(N)} := \prod_{j=1}^{k-1}\varepsilon_{i_j i_{j+1}}^{(N)}. \tag{1.5}$$

Here δ_{ij} is the Kronecker delta. For simplicity, we sometimes write $S_k^{(N)}(n_1,\dots,n_k) = S_k^{(N,N)}(n_1,\dots,n_k)$, $S_k^{(N,M)}(n) = S_k^{(N,M)}(n,\dots,n)$ and $S_k^{(N)}(n) = S_k^{(N)}(n,\dots,n)$. We note that $S_1^{(N,M)}(n) = Li_n(\omega_M)$ where $Li_n(z) := \sum_{i=1}^{\infty} z^i/i^n$ is the polylogarithm.

The aim of the paper is to establish generating function formulas for the series $S_k^{(N,M)}(n)$, and give an explicit evaluation of them in terms of Bernoulli numbers in the special case where $N = M = 2$ and n is even. It is quite remarkable that the values $S_k^{(2)}(n)$ can be fully computable; in fact, there are few examples of computable multiple L-values. In this sense, $S_k^{(N)}(n)$ seems to be a nice variant of the ordinary multiple L-values.

We will sometimes call $S_k^{(N,M)}(n_1,\dots,n_k)$ as a *partial multiple L-value* because it is indeed a partial sum of the "non-strict" multiple L-value

$$\sum_{1\le i_1\le i_2\le\dots\le i_k}\frac{\omega_M^{i_1+i_2+\dots+i_k}}{i_1^{n_1}i_2^{n_2}\dots i_k^{n_k}} = S_k^{(1,M)}(n_1,n_2,\dots,n_k).$$

In particular, $S_k^{(1)}(n_1,n_2,\dots,n_k)$ gives the non-strict multiple zeta value (see, e.g. [12]). It is also worth remarking that $\varepsilon_{ij}^{(N)} \to 1-\delta_{ij}$ as $N\to\infty$ for fixed indices i,j, so that we may regard the (strict) multiple L-values (1.3) as "limiting case" $S^{(\infty,M)}(n_1,n_2,\dots,n_k)$ of our series. We notice that our partial multiple L-value $S_k^{(N,M)}(n_1,\dots,n_k)$ is a special case of *neither* the multiple L-values (1.2) nor (1.3) since $\varepsilon_{i_1 i_2\dots i_k}^{(N)}$ does depend on *both* the differences $i_j - i_{j-1}$ of adjacent indices and the values of the indices $i_1,\dots,i_k$ themselves. However, it is not difficult to see that $S_k^{(N,M)}(n_1,\dots,n_k)$ can be expressed as a $\mathbb{Q}$-linear combination of (1.2) (or (1.3)). Thus, for fixed N and M, it may be interesting to study the structure of the subalgebra spanned by all $S_k^{(N,M)}(n_1,n_2,\dots,n_k)$ in the $\mathbb{Q}$-algebra spanned by all multiple L-values $S_k^{(1,M)}(n_1,\dots,n_k)$. We leave these problems to the future study.

We now explain the spectral-theoretic origin of our series $S_k^{(N,M)}(n_1,\dots,n_k)$. A system of differential equations defined by the operator

$$Q := \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\right) + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\left(x\frac{d}{dx}+\frac{1}{2}\right)$$

having two real parameters α,β is called the *non-commutative harmonic oscillator*. This system was first introduced and extensively studied by Parmeggiani and Wakayama [16, 17] (see also [15]). It is shown that when $\alpha,\beta>0$ and $\alpha\beta>1$, Q defines a positive, self-adjoint operator on $L^2(\mathbb{R})\otimes\mathbb{C}^2$ which has only a discrete spectrum $(0<)\lambda_1\le\lambda_2\le\dots\le\lambda_n\le\dots(\nearrow+\infty)$, and the multiplicities of the eigenvalues are uniformly bounded. In order to describe the total behavior of the spectrum, Ichinose and Wakayama [6] studied the spectral zeta function $\zeta_Q(s) := \sum_{n=1}^{\infty}\lambda_n^{-s}$ which is absolute convergent if $\mathrm{Re}(s)>1$. This is analytically continued to the whole plane $\mathbb{C}$ and gives a single-valued meromorphic function which has a simple pole at $s=1$ and 'trivial' zeros at nonpositive even integers. If $\alpha=\beta=1/\sqrt{2}$, then Q is unitarily equivalent to a couple of the (ordinary) harmonic oscillators, from which it follows that $\zeta_Q(s) = 2(2^s-1)\zeta(s)$. Thus one can regard $\zeta_Q(s)$ as a deformation of the Riemann zeta function $\zeta(s)$.

In describing the special values of the spectral zeta function $\zeta_Q(s)$, the integrals

$$J_m(n) = 2^m\int_0^1\dots\int_0^1\left(\frac{(1-x_1^4)(1-x_2^4\cdots x_m^4)}{(1-x_1^2\cdots x_m^2)^2}\right)^n\frac{dx_1\cdots dx_m}{1-x_1^2\cdots x_m^2}\quad(m=2,3,4,\dots;\ n=0,1,2,\dots)$$

and their generating functions $g_m(x) = \sum_{n=0}^{\infty}\binom{-1/2}{n}J_m(n)x^n$ play a very important role. In fact, Ichinose and Wakayama [7] calculated the first two special values $\zeta_Q(2)$ and $\zeta_Q(3)$ in terms of $g_2(x)$ and $g_3(x)$, respectively.

The higher special values $\zeta_Q(m)$ ($m \geq 4$) are also expected to be expressed by $g_m(x)$ and their generalizations (see, e.g. [13, 9, 8]). In the case where $m = 2r$ is even, $J_{2r}(n)$ is explicitly given by

$$J_{2r}(n) = \sum_{p=0}^{n} (-1)^p \binom{-\frac{1}{2}}{p}^2 \binom{n}{p} \sum_{k=0}^{r-1} \zeta\left(2r - 2k, \frac{1}{2}\right) S_{k,p},$$

where $\zeta(s, x) := \sum_{n=0}^{\infty} (n + x)^{-s}$ is the Hurwitz zeta function and

$$S_{k,p} = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq 2p} \varepsilon^{(2)}_{i_1 i_2 \dots i_k} \frac{(-1)^{i_1+i_2+\cdots+i_k}}{i_1^2 i_2^2 \dots i_k^2}.$$

Now it is immediate to see that our series $S_k^{(N,M)}(n_1, \dots, n_k)$ is a natural generalization of $S_k^{(2)}(2) = \lim_{p\to\infty} S_{k,p}$ (we give the explicit formula of $S_k^{(2)}(2)$ in Example 3.4).

It is also worth remarking that another kind of generating function $w_2(t) = \sum_{n=0}^{\infty} J_2(n)t^n$ of $J_2(n)$ is regarded as a period integral for the universal family of the elliptic curves equipped with a rational point of order 4, and satisfies a Picard-Fuchs differential equation attached to this family of curves [10].

Conventions

We recall several basic conventions on partitions and symmetric functions (for further details, see [5]).

A partition is a weakly decreasing sequence of nonnegative integers which has finitely many nonzero entries. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ ($\lambda_l \geq 1$), the sum $\lambda_1 + \cdots + \lambda_l$ of entries in λ is denoted by $|\lambda|$ and the number l of nonzero entries in λ is denoted by $\ell(\lambda)$. We write $\lambda \vdash k$ to imply $|\lambda| = k$, and say λ is a partition of k. We denote by $\emptyset$ the (only) partition of 0. To indicate a multiple of the same numbers in λ, we often write in an exponential form; Let $m_i = m_i(\lambda)$ be the number of i's in λ. We call $m_i(\lambda)$ the multiplicity of i in λ. Then, we also write $\lambda = (k^{m_k}, \dots, 2^{m_2}, 1^{m_1})$ or $\lambda = 1^{m_1}2^{m_2} \dots k^{m_k}$. For instance, $\lambda = (4, 2, 2, 2, 1, 1, 1)$ is also written as $\lambda = (4, 2^2, 1^3) = 1^3 2^2 4^1$. When all the entries of λ is even, we call λ an even partition. For a given partition $\mu = (\mu_1, \dots, \mu_l)$ and a positive integer q, we define $q\mu = (q\mu_1, \dots, q\mu_l)$. We notice that $\{\lambda \vdash 2k \,|\, \lambda\colon \text{even}\} = \{2\mu \,|\, \mu \vdash k\}$. If a given pair of two partitions λ and μ satisfies that $\lambda_i - \mu_i = 0$ or 1 for any index i, then we say λ/μ is a vertical strip. For instance, $(4, 2, 2, 1, 1, 1)/(3, 2, 1, 1)$ is a vertical strip.

Let $f(n)$ be a function on $\mathbb{N}$ and a_n a sequence. Then, for a partition λ and $q \in \mathbb{N}$, we put $f(q\lambda) := \prod_{j=1}^{\ell(\lambda)} f(q\lambda_j)$ and $a_{q\lambda} := \prod_{j=1}^{\ell(\lambda)} a_{q\lambda_j}$. For instance, $(q\lambda)! = \prod_{j=1}^{\ell(\lambda)} (q\lambda_j)!$.

Let $x_1, x_2, \dots$ be (infinitely many) variables. For each positive integer r, we respectively denote by $e_r = e_r(x_1, x_2, \dots)$ and $h_r = h_r(x_1, x_2, \dots)$ the r-th elementary and r-th complete symmetric function defined by

$$e_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}, \qquad h_r = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \dots x_{i_r}.$$

We also put $e_0 = h_0 = 1$ for convenience. Moreover, for a partition λ, we put $e_\lambda = \prod_{i\geq 1} e_{\lambda_i}$ and $h_\lambda = \prod_{i \geq 1} h_{\lambda_i}$. The generating functions of e_r and h_r are given by

$$E(t) = \sum_{r=0}^{\infty} e_r t^r = \prod_{n=1}^{\infty} (1 + x_n t), \qquad H(t) = \sum_{r=0}^{\infty} h_r t^r = \prod_{n=1}^{\infty} (1 - x_n t)^{-1}. \tag{1.6}$$

2 Generating functions

In this section, we establish generating function formulas for the series $S_k^{(N,M)}(n)$. To achieve this, we first consider a decomposition of the non-strict multiple sum $S_k^{(N,M)}(n)$ into the sum of several strict multiple sums. Notice that each increasing sequence $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k$ of k positive integers uniquely determines a

sequence $\boldsymbol{r} = (r_1, r_2, \ldots, r_l)$, which we will refer to as the *multiplicity* of the sequence $(i_1, i_2, \ldots, i_k)$, through the condition

$$\underbrace{i_1 = \cdots = i_{r_1}}_{r_1} < \underbrace{i_{r_1+1} = \cdots = i_{r_1+r_2}}_{r_2} < i_{r_1+r_2+1} = \cdots = i_{r_1+\cdots+r_{l-1}} < \underbrace{i_{r_1+\cdots+r_{l-1}+1} = \cdots = i_{r_1+\cdots+r_l}}_{r_l}.$$

Obviously, $\boldsymbol{r}$ is a permutation of a certain partition of k. We denote by $\widetilde{S}^{(N,M)}(n;\boldsymbol{r})$ the partial sum of $S^{(N,M)}(n)$ whose running indices have multiplicity $\boldsymbol{r}$, i.e.

$$\widetilde{S}^{(N,M)}(n;\boldsymbol{r}) = \sum_{j_1 < \cdots < j_l} \varepsilon^{(N)}_{\underbrace{j_1,\ldots,j_1}_{r_1},\ldots,\underbrace{j_l,\ldots,j_l}_{r_l}} \frac{\omega_M^{r_1 j_1 + \cdots + r_l j_l}}{j_1^{nr_1} \cdots j_l^{nr_l}} = \sum_{\substack{j_1 < \cdots < j_l \\ r_i > 1 \Rightarrow N | j_i}} \frac{\omega_M^{r_1 j_1 + \cdots + r_l j_l}}{j_1^{nr_1} \cdots j_l^{nr_l}}.$$

We also put

$$S^{(N,M)}(n;\emptyset) = 1, \quad S^{(N,M)}(n;\lambda) = \sum_{\boldsymbol{r} \in P(\lambda)} \widetilde{S}^{(N,M)}(n;\boldsymbol{r}) \quad (\lambda \neq \emptyset),$$

where $P(\lambda)$ denotes the set consisting of the permutations of a partition λ. It is easy to see that

$$S_k^{(N,M)}(n) = \sum_{\lambda \vdash k} S^{(N,M)}(n;\lambda). \tag{2.1}$$

To study the series $S^{(N,M)}(n;\lambda)$, we here employ another function $R^{(N,M)}(n;\mu)$ defined by

$$R^{(N,M)}(n;\emptyset) := 1, \quad R^{(N,M)}(n;\mu) := S^{(N,M)}(n;\mu_{>1}) S^{(N,M)}(n;1^{m_1(\mu)}) \qquad (\mu \neq \emptyset).$$

Here $\mu_{>1}$ denotes the partition defined by $\mu_{>1} := 2^{m_2(\mu)} 3^{m_3(\mu)} \ldots$. Fix a partition $\mu \vdash k$ and put $q = m_1(\mu)$, $p = \ell(\mu) - q$. We easily see that

$$R^{(N,M)}(n;\mu) = \sum_{\boldsymbol{r} \in P(\mu_{>1})} \widetilde{R}^{(N,M)}_{p,q}(n;\boldsymbol{r}), \qquad \widetilde{R}^{(N,M)}_{p,q}(n;\boldsymbol{r}) = \sum_{\substack{s_1 < \cdots < s_p \\ t_1 < \cdots < t_q \\ N | s_i}} \frac{\omega_M^{r_1 s_1 + \cdots + r_p s_p + t_1 + \cdots + t_q}}{s_1^{r_1 n} \cdots s_p^{r_p n} t_1^n \cdots t_q^n}.$$

In the sum $\widetilde{R}^{(N,M)}_{p,q}(n;\boldsymbol{r})$ for each $\boldsymbol{r} \in P(\mu_{>1})$, several of the running indices $t_1, \ldots, t_q$ may coincide with certain $s_1, \ldots, s_p$. To describe the situation, we introduce the following map: Put

$$I(p,q) = \left\{ (\boldsymbol{\tau}, \boldsymbol{\varepsilon}) = (\tau_0, \tau_1, \ldots, \tau_p, \varepsilon_1, \ldots, \varepsilon_p)\,;\, \tau_i \in \mathbb{Z}_{\geq 0},\ \varepsilon_i \in \{0,1\},\ \sum_{i=0}^{p} \tau_i + \sum_{i=1}^{p} \varepsilon_i = q \right\}.$$

For each element $(\boldsymbol{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) \in P(\mu_{>1}) \times I(p,q)$, we associate a new sequence $\pi_\mu(\boldsymbol{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon}))$ by

$$\pi_\mu(\boldsymbol{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) = (1^{\tau_0}, r_1 + \varepsilon_1, 1^{\tau_1}, r_2 + \varepsilon_2, \ldots, r_p + \varepsilon_p, 1^{\tau_p}).$$

Notice that there exists a partition $\lambda \vdash k$ such that $\lambda/\mu_{>1}$ is a vertical strip and $\pi_\mu(\boldsymbol{r}, (\boldsymbol{\tau}, \boldsymbol{\varepsilon})) \in P(\lambda)$. Namely, the correspondence π_μ defines a map $\pi_\mu : P(\mu_{>1}) \times I(p,q) \to \coprod_{\substack{\lambda \vdash k \\ \lambda/\mu_{>1}:\text{vertical strip}}} P(\lambda)$. Thus it follows that

$$\sum_{\boldsymbol{r} \in P(\mu_{>1})} \widetilde{R}^{(N,M)}_{p,q}(n;\boldsymbol{r}) = \sum_{\substack{\lambda \vdash k \\ \lambda/\mu_{>1}:\text{vertical strip}}} \sum_{\boldsymbol{r} \in P(\lambda)} \left| \pi_\mu^{-1}(\boldsymbol{r}) \right| \widetilde{S}^{(N,M)}(n;\boldsymbol{r}).$$

Since each $\left|\pi_\mu^{-1}(\boldsymbol{r})\right|$ depends *only* on λ, we obtain

$$R^{(N,M)}(n;\mu) = \sum_{\substack{\lambda \vdash k \\ \lambda/\mu_{>1}:\text{vertical strip}}} \left| \pi_\mu^{-1}(\lambda) \right| S^{(N,M)}(n;\lambda).$$

Next, we calculate $\left|\pi_\mu^{-1}(\lambda)\right|$. For each $a>2$, we assume that $\lambda_{i_{a1}}=\cdots=\lambda_{i_{a,d(a)}}=a$, where $d(a)=m_a(\lambda)$. Let us count the number of elements $(\boldsymbol{r},(\boldsymbol{\tau},\boldsymbol{\varepsilon}))$ in $I(p,q)$ such that $\pi_\mu(\boldsymbol{r},(\boldsymbol{\tau},\boldsymbol{\varepsilon}))=\lambda$. Notice that $\boldsymbol{\tau}$ is uniquely determined by the assumption. If $r_{i_{aj}}+\varepsilon_{i_{aj}}=\lambda_{i_{aj}}=a$, then it is possible that $(r_{i_{aj}},\varepsilon_{i_{aj}})=(a,0)$ or $(a-1,1)$, and there are exactly $\binom{m_a(\lambda)}{m_a(\lambda;\mu)}$ ways of the choice of i_{aj} such that $(r_{i_{aj}},\varepsilon_{i_{aj}})=(a,0)$, where $m_i(\lambda;\mu)=|\{j\,;\,\lambda_j=\mu_j=i\}|$. (Remark that $m_2(\lambda;\mu)=m_2(\lambda)$.) Thus we have $\left|\pi_\mu^{-1}(\lambda)\right|=\prod_{i>2}\binom{m_a(\lambda)}{m_a(\lambda;\mu)}$. If μ is an *even* partition and $\mu/\lambda_{>1}$ is a vertical strip, then $m_i(\lambda;\mu)=m_i(\lambda)$ (if i is even) or 0 (if i is odd) by definition, and hence $\left|\pi_\mu^{-1}(\lambda)\right|=1$. Consequently, we get the following lemma.

Lemma 2.1. *For each $\mu\vdash k$, it holds that*

$$R^{(N,M)}(n;\mu)=\sum_{\substack{\lambda\vdash k\\ \lambda/\mu_{>1}:\text{vertical strip}}}\prod_{i>2}\binom{m_i(\lambda)}{m_i(\lambda;\mu)}S^{(N,M)}(n;\lambda), \tag{2.2}$$

where $m_i(\lambda;\mu)=|\{j\,;\,\lambda_j=\mu_j=i\}|$. In particular, if μ is even, then

$$R^{(N,M)}(n;\mu)=\sum_{\substack{\lambda\vdash k\\ \lambda/\mu_{>1}:\text{vertical strip}}}S^{(N,M)}(n;\lambda). \tag{2.3}$$

□

Lemma 2.2. *For any $\lambda\vdash k$, there uniquely exists $\mu\vdash k$ such that $\mu_{>1}$ is even and $\lambda/\mu_{>1}$ is a vertical strip.*

Proof. It is immediate to see that $\mu=1^{m_1(\lambda)+m_3(\lambda)+m_5(\lambda)+\cdots}2^{m_2(\lambda)+m_3(\lambda)}4^{m_4(\lambda)+m_5(\lambda)}6^{m_6(\lambda)+m_7(\lambda)}\cdots\vdash k$ is a unique partition which satisfies all the desired conditions. □

By Lemmas 2.1 and 2.2, we readily obtain the

Lemma 2.3. *Let $U_d^{(N,M)}(n):=\sum_{\mu\vdash d}S^{(N,M)}(n;2\mu)$. Then it holds that*

$$S_k^{(N,M)}(n)=\sum_{\lambda\vdash k}S^{(N,M)}(n;\lambda)=\sum_{\substack{\mu\vdash k\\ \mu_{>1}:\text{even}}}R^{(N,M)}(n;\mu)=\sum_{0\le 2d\le k}S^{(N,M)}(n;1^{k-2d})U_d^{(N,M)}(n). \tag{2.4}$$

□

We next study the generating function of $S_k^{(N,M)}(n)$. For this purpose, the following formula, which is obtained by the canonical product expression of the gamma function, is useful.

Lemma 2.4. *For $a_i,b_i\in\mathbb{C}$ satisfying $\sum_{i=1}^l a_i=\sum_{i=1}^l b_i$, the equality*

$$\prod_{m=k}^{\infty}\prod_{j=1}^{l}\frac{m+a_j}{m+b_j}=\prod_{j=1}^{l}\frac{\Gamma(k+b_j)}{\Gamma(k+a_j)} \tag{2.5}$$

holds for any integer k. □

Lemma 2.5. *The generating function of $U_d^{(N,M)}(n)$ is given by*

$$\mathcal{H}^{(N,M)}(n;x):=\sum_{d=0}^{\infty}U_d^{(N,M)}(n)x^{2nd}=\prod_{k=1}^{M}\prod_{j=0}^{2n-1}\frac{\Gamma\Big(\frac{1}{M}\Big(k-\frac{1}{N}\omega_{2n}^{j}\omega_{Mn}^{kN}x\Big)\Big)}{\Gamma\big(\frac{k}{M}\big)}. \tag{2.6}$$

Proof. We notice that

$$U_d^{(N,M)}(n)=\sum_{\mu\vdash d}S^{(N)}(n;2\mu)=h_d\Big(\frac{\omega_M^{2N}}{N^{2n}},\frac{\omega_M^{4N}}{(2N)^{2n}},\frac{\omega_M^{6N}}{(3N)^{2n}},\ldots\Big)$$

since the complete symmetric function h_d is the sum of *all* monomials of degree d. Therefore, by specializing $x_m = \omega_M^{2mN}/(Nm)^{2n}$ and $t = x^{2n}$ in the generating function $H(t)$ in (1.6), we obtain

$$\begin{aligned}\mathcal{H}^{(N,M)}(n;x) &= \prod_{m=1}^{\infty}\left(1-\frac{\omega_M^{2mN}}{(Nm)^{2n}}x^{2n}\right)^{-1} = \prod_{m=0}^{\infty}\prod_{k=1}^{M}\left\{1-\left(\frac{\omega_{Mn}^{kN}x}{N(Mm+k)}\right)^{2n}\right\}^{-1}\\ &= \prod_{m=0}^{\infty}\prod_{k=1}^{M}\prod_{j=0}^{2n-1}\left(1-\omega_{2n}^{j}\frac{\omega_{Mn}^{kN}x}{N(Mm+k)}\right)^{-1} = \prod_{m=0}^{\infty}\prod_{k=1}^{M}\prod_{j=0}^{2n-1}\frac{m+\frac{k}{M}-\frac{\omega_{2n}^{j}\omega_{Mn}^{kN}x}{MN}}{m+\frac{k}{M}}.\end{aligned}$$

Applying Lemma 2.4 to the equation above, we have (2.6). □

Lemma 2.6. *The generating function of $S^{(N,M)}(n;1^r)$ is given by*

$$\mathcal{E}^{(M)}(n;x) := \sum_{r=0}^{\infty} S^{(N,M)}(n;1^r)x^{nr} = \prod_{k=1}^{M}\prod_{j=0}^{n-1}\frac{\Gamma\left(\frac{k}{M}\right)}{\Gamma\left(\frac{1}{M}\left(k-\omega_{2n}^{2j-1}\omega_{Mn}^{k}x\right)\right)}. \tag{2.7}$$

Proof. We notice that

$$S^{(N)}(n;1^r) = e_r\left(\frac{\omega_M}{1^n}, \frac{\omega_M^2}{2^n}, \frac{\omega_M^3}{3^n}, \ldots\right).$$

Hence, if we specialize $x_m = \omega_M^m/m^n$ and set $t = x^n$ in the generating function $E(t)$ in (1.6), then we obtain the lemma by a similar calculation as in the case of $\mathcal{H}^{(N,M)}(n;x)$. □

Now, we obtain the following

Theorem 2.7. *The generating function of $S_k^{(N,M)}(n)$ is given by*

$$\mathcal{S}^{(N,M)}(n;x) := \sum_{k=0}^{\infty} S_k^{(N)}(n)x^{nk} = \prod_{k=1}^{M}\frac{\prod_{j=0}^{2n-1}\Gamma\left(\frac{1}{M}\left(k-\frac{1}{N}\omega_{2n}^{j}\omega_{Mn}^{kN}x\right)\right)}{\Gamma\left(\frac{k}{M}\right)^n\prod_{j=0}^{n-1}\Gamma\left(\frac{1}{M}\left(k-\omega_{2n}^{2j-1}\omega_{Mn}^{k}x\right)\right)}. \tag{2.8}$$

Proof. From the equation (2.4), it is clear that $\mathcal{S}^{(N,M)}(n;x) = \mathcal{H}^{(N,M)}(n;x)\mathcal{E}^{(M)}(n;x)$. Hence one immediately obtains the formula (2.8) from (2.6) and (2.7). □

If $M \mid N$, then, using the Gauss-Legendre formula of the gamma function, we have the following reduced formulas:

$$\mathcal{H}^{(N,M)}(n;x) = \prod_{j=0}^{2n-1}\Gamma\left(1-\frac{\omega_{2n}^{j}x}{N}\right), \tag{2.9}$$

$$\mathcal{S}^{(N,M)}(n;x) = \prod_{k=1}^{M}\frac{\Gamma\left(\frac{k}{M}\right)^n\prod_{j=0}^{2n-1}\Gamma\left(1-\frac{\omega_{2n}^{j}x}{N}\right)}{\prod_{j=0}^{n-1}\Gamma\left(\frac{1}{M}\left(k-\omega_{2n}^{2j-1}\omega_{Mn}^{k}x\right)\right)}. \tag{2.10}$$

Notice that $\mathcal{E}^{(M)}(n;x)$ depends *only* on M.

3 Partial alternating multiple zeta values

In this section, we concentrate on the special case where $N = M = 2$. From the definition, the sums $S_k(n) := S_k^{(2,2)}(n)$ in this case may be called *partial alternating multiple zeta values*. From (2.10), we have

$$\mathcal{S}(n;x) := \mathcal{S}^{(2)}(n;x) = \frac{\Gamma\left(\frac{1}{2}\right)^n\prod_{j=0}^{2n-1}\Gamma\left(1-\frac{x}{2}\omega_{2n}^{j}\right)}{\prod_{j=0}^{n-1}\Gamma\left(\frac{1}{2}-\frac{x}{2}\omega_n^{j}\right)\Gamma\left(1-\frac{x}{2}\omega_n^{j}\omega_{2n}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)^n\prod_{j=0}^{n-1}\Gamma\left(1-\frac{x}{2}\omega_n^{j}\right)}{\prod_{j=0}^{n-1}\Gamma\left(\frac{1}{2}-\frac{x}{2}\omega_n^{j}\right)}.$$

Furthermore, using the duplication formula $\Gamma(2a)\Gamma(1/2) = 2^{2a-1}\Gamma(a)\Gamma(1/2+a)$ with $a = -x\omega_n^j/2$ and the equation $\sum_{j=0}^{n-1}\omega_n^j = \delta_{n,1}$, we see

$$\mathcal{S}(n;x) = \frac{\Gamma\big(\frac{1}{2}\big)^n \prod_{j=0}^{n-1}\Gamma\big(1-\frac{x}{2}\omega_n^j\big)}{\prod_{j=0}^{n-1}\Gamma(-x\omega_n^j)\Gamma\big(\frac{1}{2}\big)\,2^{x\omega_n^j+1}\Gamma\Big(-\frac{x}{2}\omega_n^j\Big)^{-1}} = 2^{-x\delta_{n,1}}\prod_{j=0}^{n-1}\frac{\Gamma\big(1-\frac{x}{2}\omega_n^j\big)^2}{\Gamma\Big(1-x\omega_n^j\Big)}. \tag{3.1}$$

For $m \geq 0$, define the sequence $\{A^\bullet(m)\}_{m\geq 0}$ by $A^\bullet(0) := 1$, $A^\bullet(1) := 0$ and

$$A^\bullet(m) := \sum_{a=1}^{m-1}\zeta_a^\bullet(\underbrace{1,\dots,1}_{a-1}, m-a+1) \qquad (m \geq 2).$$

Namely, $A^\bullet(m)$ $(m \geq 2)$ denotes the sum of multiple zeta values of weight m and height 1. It is known that $A^\bullet(m)$ can be expressed as a polynomial in $\zeta(2), \zeta(3), \dots$ and $\zeta(m)$ with rational coefficients (see [14]). For example, we have $A^\bullet(3) = \zeta(3) + \zeta_2^\bullet(1,2) = 2\zeta(3)$ since $\zeta_2^\bullet(1,2) = \zeta(3)$, which is due to Euler. Further, we put

$$A_n^\bullet(m) := \sum_{\substack{m_1,\dots,m_n\geq 0\\ m_1+\cdots+m_n=m}} A^\bullet(m_1)\cdots A^\bullet(m_n), \qquad Z_n(k) := \sum_{\substack{\mu\vdash k\\ \mu_{\ell(\mu)}>\delta_{n,1}}}\frac{\nu(n\mu)}{z_\mu}\zeta(n\mu),$$

where $\nu(x) := 2^{1-x} - 1$ and $z_\mu := \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)!$. Note that $A_1^\bullet(m) = A^\bullet(m)$. Then, we get the following expressions of the values $S_k(n) = S_k^{(2)}(n)$.

Theorem 3.1. (i) *If $n = 1$, then it holds that*

$$S_k(1) = \sum_{m=0}^{k}\frac{(-\log 2)^{k-m}}{(k-m)!2^m}A^\bullet(m) = \sum_{m=0}^{k}\frac{(-\log 2)^{k-m}}{(k-m)!}Z_1(m) \in \mathbb{Q}[\log 2, \zeta(2), \zeta(3), \dots, \zeta(k)]. \tag{3.2}$$

(ii) *If $n \geq 2$, then it holds that*

$$S_k(n) = \frac{1}{2^{nk}}A_n^\bullet(nk) = Z_n(k) \in \mathbb{Q}[\zeta(n), \zeta(2n), \dots, \zeta(kn)]. \tag{3.3}$$

Proof. From the generating function (3.1), it is sufficient to show that

$$\prod_{j=0}^{n-1}\frac{\Gamma\big(1-\frac{x}{2}\omega_n^j\big)^2}{\Gamma\Big(1-x\omega_n^j\Big)} = \sum_{m=0}^{\infty}A_n^\bullet(nm)\left(\frac{x}{2}\right)^{nm} = \sum_{m=0}^{\infty}Z_n(m)x^{nm}. \tag{3.4}$$

To prove this, we recall the identity (see [1, 4])

$$\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} = 1 - \sum_{a,b=1}^{\infty}\zeta_a^\bullet(\underbrace{1,\dots,1}_{a-1}, b+1)X^aY^b = \exp\left(\sum_{m=2}^{\infty}\frac{X^m+Y^m-(X+Y)^m}{m}\zeta(m)\right). \tag{3.5}$$

Putting $X = Y = x\omega_n^j/2$ and writing $a + b = m$ in the middle term in (3.5), we have

$$\frac{\Gamma\big(1-\frac{x}{2}\omega_n^j\big)^2}{\Gamma\Big(1-x\omega_n^j\Big)} = \sum_{m=0}^{\infty}A^\bullet(m)\left(\frac{x\omega_n^j}{2}\right)^m = \exp\left(\sum_{m=2}^{\infty}\frac{\nu(m)}{m}\zeta(m)(\omega_n^j x)^m\right).$$

Then, taking the product $\prod_{j=0}^{n-1}$ of this equation, one sees that

$$\prod_{j=0}^{n-1}\frac{\Gamma\big(1-\frac{x}{2}\omega_n^j\big)^2}{\Gamma\Big(1-x\omega_n^j\Big)} = \sum_{m=0}^{\infty}A_n^\bullet(m)\left(\frac{x\omega_n^j}{2}\right)^m = \exp\left(\sum_{\substack{m=1\\ nm\geq 2}}^{\infty}\frac{\nu(nm)}{m}\zeta(nm)x^{nm}\right) \tag{3.6}$$

because $\sum_{j=0}^{n-1}\omega_n^{jm}=n$ if $n\mid m$ and 0 otherwise. Here, the rightmost-hand side of (3.6) can be written as

$$\begin{aligned}\prod_{\substack{m=1\\ nm\geq 2}}^{\infty}\exp\left(\frac{\nu(nm)}{m}\zeta(nm)x^{nm}\right)&=\prod_{\substack{m=1\\ nm\geq 2}}^{\infty}\sum_{l_m=0}^{\infty}\frac{1}{l_m!}\left(\frac{\nu(nm)}{m}\zeta(nm)x^{nm}\right)^{l_m}\\&=\begin{cases}\displaystyle\sum_{l_2,l_3,\ldots=0}^{\infty}\frac{\nu(2)^{l_2}\nu(3)^{l_3}\cdots}{(2^{l_2}3^{l_3}\cdots)(l_2!l_3!\cdots)}\left(\zeta(2)^{l_1}\zeta(3)^{l_3}\cdots\right)x^{2l_2+3l_3+\cdots} & (n=1)\\ \displaystyle\sum_{l_1,l_2,\ldots=0}^{\infty}\frac{\nu(n)^{l_1}\nu(2n)^{l_2}\cdots}{(1^{l_1}2^{l_2}\cdots)(l_1!l_2!\cdots)}\left(\zeta(n)^{l_1}\zeta(2n)^{l_2}\cdots\right)x^{l_1+2l_2+\cdots} & (n\geq 2)\end{cases}\\&=\sum_{m=0}^{\infty}\left\{\sum_{\substack{\mu\vdash m\\ \mu_{\ell(\mu)}>\delta_{n,1}}}\frac{\nu(n\mu)}{z_\mu}\zeta(n\mu)\right\}x^{nm}=\sum_{m=0}^{\infty}Z_n(m)x^{nm}.\end{aligned}$$

Note that, from the second equality in (3.6), this shows that $A_n^{\bullet}(m)=0$ if $n\nmid m$. Therefore, one can actually obtain the equations (3.4). This completes the proof of the theorem. □

Example 3.2. We have

$$S_1(1)=-\log 2,\quad S_2(1)=\frac{(\log 2)^2}{2}-\frac{\zeta(2)}{4},\quad S_3(1)=-\frac{(\log 2)^3}{6}+\frac{\log 2}{4}\zeta(2)-\frac{1}{4}\zeta(3),$$

and

$$S_1(3)=-\frac{3}{4}\zeta(3),\quad S_2(3)=-\frac{31}{64}\zeta(6)+\frac{9}{32}\zeta(3)^2,\quad S_3(3)=-\frac{255}{768}\zeta(9)+\frac{93}{128}\zeta(6)\zeta(3)-\frac{27}{384}\zeta(3)^3.$$

If one further assumes that n is even, then one can obtain the following various expressions.

Theorem 3.3. *It holds that*

$$\begin{aligned}S_k(2n)&=(-\pi^2)^{nk}\sum_{\substack{m_1,\ldots,m_n\geq 0\\ m_1+\cdots+m_n=nk}}\omega_n^{m_1+2m_2+\cdots+nm_n}\frac{B_{2m_1}}{(2m_1)!}\cdots\frac{B_{2m_n}}{(2m_n)!} &(3.7)\\&=(-\pi^2)^{nk}\sum_{\substack{\lambda\vdash nk\\ \ell(\lambda)\leq n}}\langle p_n\circ h_k,\,m_\lambda\rangle\frac{B_{2\lambda}}{(2\lambda)!} &(3.8)\\&=(-\pi^2)^{nk}\sum_{\mu\vdash k}\frac{\widetilde{\nu}(2n\mu)}{z_\mu}\frac{B_{2n\mu}}{(2n\mu)!}, &(3.9)\end{aligned}$$

where $\widetilde{\nu}(x):=2^{x-1}-1$, p_n is the n-th power-sum symmetric function, m_λ the monomial symmetric function for λ, $\circ$ the plethysm, and $\langle\cdot,\cdot\rangle$ the standard scalar product in the ring of symmetric functions defined by $\langle h_\lambda, m_\mu\rangle=\delta_{\lambda\mu}$ with $\delta_{\lambda\mu}$ being the Kronecker delta (see [5] for detail).

Proof. If we apply the reflection formula for the gamma function in (3.1), then we have

$$\mathcal{S}(2n;x)=\prod_{j=1}^{n}\frac{\Gamma\Big(1-\frac{\omega_{2n}^jx}{2}\Big)^2\Gamma\Big(1+\frac{\omega_{2n}^jx}{2}\Big)^2}{\Gamma\Big(1-\omega_{2n}^jx\Big)\Gamma\Big(1+\omega_{2n}^jx\Big)}=\prod_{j=1}^{n}\frac{\pi x\omega_{2n}^j}{2}\cot\frac{\omega_{2n}^j\pi x}{2}=\prod_{j=1}^{n}\sum_{m=0}^{\infty}\frac{(-\omega_n^j)^mB_{2m}\pi^{2m}}{(2m)!}x^{2m},\tag{3.10}$$

from which we immediately obtain (3.7). Next, it readily follows from (3.7) that

$$S_k^{(2n)}=(-\pi^2)^{nk}\sum_{\substack{\lambda\vdash nk\\ \ell(\lambda)\leq n}}m_\lambda(1,\omega_n,\ldots,\omega_n^{n-1},0,\ldots)\frac{B_{2\lambda}}{(2\lambda)!}.$$

Thus we should calculate $m_\lambda(1,\omega_n,\dots,\omega_n^{n-1},0,\dots)$. Let us recall the expansion formula (see, e.g. [5])

$$\prod_{i,j\geq 1}\frac{1}{1-x_iy_j}=\sum_\lambda h_\lambda(x)m_\lambda(y). \tag{3.11}$$

If we set $y_j=\omega_n^{j-1}$ for $j=1,2,\dots,n$ and $y_j=0$ for $j>n$ in (3.11), then we have

$$\sum_{\ell(\lambda)\leq n}h_\lambda(x)m_\lambda(1,\omega_n,\dots,\omega_n^{n-1},0,\dots)=\prod_{i\geq 1}\frac{1}{1-x_i^n}=\sum_{k=0}^{\infty}h_k(x_1^n,x_2^n,\dots)=\sum_{k=0}^{\infty}p_n\circ h_k. \tag{3.12}$$

By taking the terms of homogeneous degree nk in (3.12), we have

$$\sum_{\substack{\lambda\vdash nk\\ \ell(\lambda)\leq n}}h_\lambda m_\lambda(1,\omega_n,\dots,\omega_n^{n-1},0,\dots)=p_n\circ h_k \tag{3.13}$$

for each k. Hence we get $m_\lambda(1,\omega_n,\dots,\omega_n^{n-1},0,\dots)=\langle p_n\circ h_k,\, m_\lambda\rangle$, which readily implies (3.8). The equation (3.9) follows immediately from (3.3) together with the classical result $\zeta(2m)=(-1)^{m-1}2^{2m-1}B_{2m}\pi^{2m}/(2m)!$ due to Euler. This completes the proof. $\square$

Example 3.4. From the equation (3.7), we have

$$S_k(2)=\frac{(-1)^kB_{2k}}{(2k)!}\pi^{2k}=-\frac{\zeta(2k)}{2^{2k-1}},\qquad S_k(4)=\left\{\sum_{m=0}^{2k}(-1)^m\frac{B_{2m}B_{4k-2m}}{(2m)!(4k-2m)!}\right\}\pi^{4k}.$$

See [19] for a similar discussion on the multiple Dirichlet L-values.

Remark 3.5. It is remarkable that $S_k(2)=S_k^{(2)}(2)$ can be reduced as above. We recall that $S_k^{(2)}(2)$ is closely related to the special value $\zeta_Q(2)$ of the spectral zeta function. Can one explain the simplicity (or "exact solvability") of $S_k^{(2)}(2)$ by, for instance, the existence of the Picard-Fuchs differential equation for $w_2(t)$?

Remark 3.6. Let us give an example of the partial alternating *double* zeta value with distinct indices:

$$S_2^{(2)}(1,2k)=(k+1)S_1^{(2)}(2k+1)+2(1-2^{-2k})\zeta(2k)\log 2-\sum_{p=1}^{k-1}S_1^{(2)}(2p+1)\zeta(2k-2p),$$

$$S_2^{(2)}(2k,1)=-kS_1^{(2)}(2k+1)-\zeta(2k)\log 2+\sum_{p=1}^{k-1}S_1^{(2)}(2p+1)\zeta(2k-2p).$$

Notice that $S_1^{(2)}(n)=(2^{1-n}-1)\zeta(n)$ for $n\geq 2$. This is regarded as an analogue of Euler's formula $\zeta_2^{\bullet}(1,2k)=k\zeta(2k+1)-\frac{1}{2}\sum_{p=2}^{2k-1}\zeta(p)\zeta(2k-p+1)$. See also [3] for related calculations.

Acknowledgement. The authors would like to thank Professor Masato Wakayama for valuable comments.

References

[1] K. Aomoto, "Special values of hyperlogarithms and linear difference schemes", *Illinois J. Math.* **34** (1990), no.2, 191–216.

[2] T. Arakawa and M. Kaneko, "On multiple L-values", *J. Math. Soc. Japan* **56** (2004), no.4, 967–991.

[3] J. M. Borwein, I. J. Zucker and J. Boersma, "The evaluation of character Euler double sums", *Ramanujan J.* (2008), available online: `DOI 10.1007/s11139-007-9083-z`.

[4] V. G. Drinfel'd, "On quasitriangular quasi-Hopf algebras and a group closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$", *Leningrad Math. J.* **2** (1991), 829–860.

[5] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, Second Edition*, Oxford Univ. Press, 1995.

[6] T. Ichinose and M. Wakayama, "Zeta functions for the spectrum of the non-commutative harmonic oscillators", *Commun. Math. Phys.* **258** (2005), 697–739.

[7] T. Ichinose and M. Wakayama, "Special values of the spectral zeta function of the non-commutative harmonic oscillator and confluent Heun equations", *Kyushu J. Math.* **59** (2005), 39–100.

[8] K. Kimoto, "Higher Apéry-like numbers arising from special values of the spectral zeta function for the non-commutative harmonic oscillator", in preparation.

[9] K. Kimoto and M. Wakayama, "Apéry-like numbers arising from special values of spectral zeta functions for non-commutative harmonic oscillators", *Kyushu J. Math.* **60** (2006), 383–404.

[10] K. Kimoto and M. Wakayama, "Elliptic curves arising from the spectral zeta function for non-commutative harmonic oscillators and $\Gamma_0(4)$-modular forms", pp. 201–218 in *The Conference on L-Functions*, World Sci. Publ., Hackensack, NJ, 2007.

[11] M. Kontsevich and D. Zagier, "Periods", pp. 771–808 in *Mathematics unlimited—2001 and beyond*, Springer, Berlin, 2001.

[12] S. Muneta, "On some explicit evaluations of multiple zeta-star values", arXiv: `0710.3219`.

[13] H. Ochiai, "A special value of the spectral zeta function of the non-commutative harmonic oscillators", *Ramanujan J.* **15** (2008), no.1, 31–36.

[14] Y. Ohno and D. Zagier, "Multiple zeta values of fixed weight, depth, and height", *Indag. Math.* **12** (2001), 483–487.

[15] A. Parmeggiani, *Introduction to the spectral theory of non-commutative harmonic oscillators*, COE Lecture Note. Kyushu University, The 21st Century COE Program "DMHF", Fukuoka, 2008.

[16] A. Parmeggiani and M. Wakayama, "Oscillator representations and systems of ordinary differential equations", *Proc. Natl. Acad. Sci. USA* **98** (2001), 26–30.

[17] A. Parmeggiani and M. Wakayama, "Non-commutative harmonic oscillators-I, II, Corrigenda and remarks to I", *Forum. Math.* **14** (2002), 539–604, 669–690, ibid **15** (2003), 955–963.

[18] V. S. Varadarajan, "Euler and his work on infinite series", *Bull. Amer. Math. Soc.* (*New Series*) **44** (2007), no.4, 515–539.

[19] Y. Yamasaki, "Evaluations of multiple Dirichlet L-values via symmetric functions", arXiv: `0712.1639`.

[20] D. Zagier, "Values of zeta functions and their applications", pp. 497–512 in *First European Congress of Mathematics, Vol. II* (*Paris, 1992*), Progr. Math., 120, Birkhäuser, Basel, 1994.

Kazufumi KIMOTO
Department of Mathematical Sciences, University of the Ryukyus
Senbaru, Nishihara, Okinawa 903-0231, Japan
`kimoto@math.u-ryukyu.ac.jp`

Yoshinori YAMASAKI
Faculty of Mathematics, Kyushu University
Hakozaki, Fukuoka 812-8581, Japan
`yamasaki@math.kyushu-u.ac.jp`